Quest One: Factors and Primes in Dimensions of Rectangles

In ancient Greece mathematicians used geometric terms to explain number relationships such as prime numbers. While there are an infinite number of prime numbers, we have no way to predict which numbers are prime.

Use rectangles to explore prime numbers

Materials needed:

- Grid paper or square tiles
- Paper for recording

A rectangle is different if it shows two distinct arrays (length or width) or arrangements. If a rectangle is rotated, it is merely a different orientation of the same rectangle and does not count as a separate rectangle.

Draw or make as many different shaped rectangles as possible using the number of squares that represent each number from 1 to 30 where each square represents a unit in length or width. For example, how many different rectangles can you make with 1 square? How many different rectangles can you make with 2 squares? How many different rectangles can you make with 3 squares?

Puzzler

Replace all *n's* with the same digit to complete the addition problem.

	3	5	n
	1	2	3
+	5	3	2
1	n	n	5

n = ______________ ?

A. On your own paper, make a table with the following headings: number of squares, factors, possible dimensions, and number of rectangles. Build or draw as many different rectangles as possible for each number of squares 1–30. Record the dimensions in your table.

B. Now add to your table the number of all the different dimensions of rectangles you can make for each number 1–30. The only column in your table still empty is the one with the factors.

Discovery 1: Expressing Prime Numbers

We can express numbers as a product of two factors. For example, the number 6 can be expressed as the product 1 x 6 or 2 x 3. Therefore, for the number 6, rectangles of 1 x 6 or2 x 3 can be built, which means that 1 x 6 and 2 x 3 are factor pairs for the number 6. The number 6, then, has four factors: 1, 6, 2, and 3. What did you find out about the number of factors for the numbers 1–30? Fill out this empty column in your table.

Closer Look

1. A number is considered prime if its only factors are 1 and itself. Examples of prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29. Only one unique rectangle can be built with numbers that have only 1 and itself as factors. All numbers with more than two factors are composite numbers. The number 1 is neither prime nor composite because it has only one distinct factor, which is 1.

- ☐ Find all the prime numbers between 30 and 100.

2. The number 1 has one factor, the numbers 4, 9, and 25 have three factors, and the number 16 has five factors. One of your rectangular arrays for each of these numbers is a square. These numbers are known as square numbers.

Find all other square numbers between 30 and 200. Remember that one of the rectangles will be a square. List each number and all factors for each number.

For each number listed, what do you notice about the number of factors? Why is this true?

Just For Fun

The product of any number multiplied by itself is called the *square* of the number, and the number that was multiplied to it is called a *square root*. For example, 4 x 4 = 16 so 16 is the square of the number 4 and 4 is a square root of 16; it is often written as "four squared" or $4^2 = 16$. The inverse (or opposite) operation of squaring a number is finding a square root of a number. To find a square root of a number, ask yourself what number times itself will yield that number. For example, if you want to know a square root of nine, think of a number that, when multiplied by itself, equals 9. Since 3 x 3 = 9, the square root of 9 is 3. We write this as $\sqrt{9} = 3$ because this is the symbol for the principal (or positive) square root, known as the radical. **Use a calculator to explore square roots of square numbers.**

4. What number do you get when you find a square root of 32? How is that answer different from square roots of square numbers?

Negative numbers are opposites of positive numbers. A negative number times a negative yields a positive number. Explore more square roots.

5. What is another square root of 9?
6. Find two square roots for 16.

When we multiply $\sqrt{4} \times \sqrt{4}$, the answer is 4 because 2 x 2 = 4. Another way to work this problem is to multiply the numbers under the radicals to form a new square root problem: $\sqrt{4} \times \sqrt{4} = \sqrt{4 \times 4} = \sqrt{16} = 4$. If we use this method, we see $\sqrt{7} \times \sqrt{7} = 7$ because $\sqrt{7} \times \sqrt{7} = \sqrt{7 \times 7} = \sqrt{49} = 7$. Do the following problems:

7. $\sqrt{9} \times \sqrt{9} =$
8. $\sqrt{8} \times \sqrt{8} =$
9. $\sqrt{25} \times \sqrt{36} =$

Quest Two The Sieve of Eratosthenes

Eratosthenes was a Greek who lived over two thousand years ago. At one time in his life he was the chief librarian at the great library in Alexandria, Egypt. Eratosthenes is known for a method of identifying prime numbers called the Sieve of Eratosthenes.

A *sieve* is a type of strainer or a colander. You might use a sieve for draining water after cooking pasta, but the sieve catches or keeps the pasta. The method by Eratosthenes can be called a sieve because it "holds" the prime numbers.

Puzzler

I have twice as many nickels as dimes.
I have 12 more nickels than dimes.
How much money do I have?

Use the hundred chart and follow the method used by Eratosthenes. All numbers except primes will fall through the sieve.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

- ☐ The number 1 has only one distinct factor—it is not prime. **Cross it out.**
- ☐ The number 2 is a prime number. **Circle it.**
- ☐ Any multiple of 2 that is greater than 2 has 2 as a factor. We know it must have 1, 2, and itself as factors. So, all multiples of 2 greater than 2 have more than two factors—they are not prime. **Cross them out.**
- ☐ The next number not crossed out is 3—it is prime. **Circle it.** 1, 3, and the number itself will be factors of all multiples of 3 greater than 3, so cross out multiples of 3 greater than 3.
- ☐ The next number not crossed out is 5—it is prime. **Circle it.** Cross out all multiples of 5 greater than 5 since these numbers will have 1, 5, and the number itself as factors.
- ☐ **Continue this process.**
- ☐ At what point can you discontinue this process?

Discovery 2: Measuring the Earth

Eratosthenes successfully measured the circumference of the earth on June 21, in Alexandria. He measured a midday shadow and calculated the circumference of the earth to be 252,000 stades (24,662 miles).

- ☐ What is the accepted value of the circumference of the earth today? In what year did Eratosthenes first measure the circumference of the earth? How far was his measurement from the accepted measurement of today? Research the answers to these questions.

Closer Look

1. Explore where prime numbers fall in relationship to numbers that are multiples of 4.

- ☐ List the multiples of 4 from 1 to 60 in a column. Compare these numbers to the primes that are less than 60.

2. Now we are going to find a large prime number. Each number has to be tested to see if it has exactly two factors: 1 and itself (testing larger numbers can take a long time).

- ☐ What is the largest prime number you can find? Write your number. Explain why you think the number is prime; use words and numbers.

3. An interesting pattern of primes is their relationship to multiples of 6.

- ☐ Make a column of multiples of 6–100. Then, look at your list of primes to see if you can make a connection between primes and multiples of 6.

4. A perfect number is exactly equal to the sum of all of its factors, not counting itself. For example, the factors of 6 are 1, 2, 3, and 6 (itself). If you add 1 + 2 + 3, the sum is 6. Therefore 6 is a perfect number.

Mersenne Primes

Over two thousand years ago, Euclid concluded that the list of prime numbers is infinite. Throughout history, many mathematicians have looked for formulas to find prime numbers. A French mathematician, Marin Mersenne, proposed a formula for finding prime numbers.

Try his formula $2^n - 1$. Here n is an exponent. An exponent means the bottom number has to be multiplied by itself the number of times given by the value of the exponent (here, n times).

- ☐ Let n equal 1, 2, 3, 4, 5, and 6. That is, plug each of 1, 2, 3, 4, 5, and 6 into the formula $2^n - 1$ to get a list of numbers. For example, if 1 is plugged in for n, the result is $2^1 - 1 = 1$.
  - What prime numbers did you get?
  - What might you tell someone about Mersenne's formula?

Just For Fun: Goldbach's Conjecture

Goldbach's Conjecture states that any even integer greater than 2 can be represented as the sum of two prime numbers. What is a conjecture? Look it up in a dictionary.

1. Investigate Goldbach's conjecture by finding sums for 12 even integers using prime numbers as addends; include some examples that are greater than 20 (remember: 2 is the only even prime number). Record your results.
   - ☐ Example: 8 = 5 + 3
   - ☐ Do you notice any pattern in the prime numbers you used to investigate Goldbach's conjecture? Can you think of another way to write Goldbach's Conjecture?

More Information

If you would like to learn more about Eratosthenes, read his biography *The Librarian who Measured the Earth*, by Kathryn Lasky. You can also check out the following sites on the Internet, some of which provide interactive demonstrations.

- The Prime Pages
  www.utm.edu/research/primes
- MacTutor History of Mathematics: Eratosthenes of Cyrene
  www-history.mcs.st-andrews.ac.uk/Biographies/Eratosthenes.html
- The University of Utah: Eratosthenes of Cyrene
  www.math.utah.edu/~pa/Eratosthenes.html
- Faust Gymnasium-Staufen: Eratosthenes Sieve
  www.fgs.snbh.schule-bw.de/mhb/eratosiv.htm